Super-resolving Single Radar Target with an Exact and Simple Formula

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***Abstract*—Ranging using an LFMCW radar (Linear Frequency Modulated Continuous Wave) amounts to estimating frequency from the radar signal. However, the ranging accuracy is nominally limited to a resolution that is equivalent to the frequency resolution of the DFT. In this paper we break this limit using a novel formula for single frequency estimation. This formula is simple yet exact, and is able to handle the damping effect which is prevalent in real-world signals. We validate our formula on simulations and on real radar measurements. Our formula is very fast, and is shown to provide sub-mm ranging accuracy using the frequency estimation, and a μm-level accuracy using the associated estimation of the phase difference.**

***Index Terms*—Frequency estimation, Radar signal processing, Super-resolution, Ranging accuracy.**

I. Introduction

The proliferation of low-cost and robust radar sensors has propelled a variety of applications such as vital sign monitoring of human beings [1] and autonomous driving [2]. Linear frequency modulated continuous wave (LFMCW) radar is one of the most common radars due to its high signal-to-noise ratio (SNR) and ranging resolution [3].

LFMCW Radar and Frequency Estimation: Ranging via LFMCW radar amounts to estimating frequency from a sinusoidal signal [3]. The radar transmitter sends a chirp, i.e., a signal with linearly increasing frequency $f(t) = f_c + tB/T_c$ where B is the sweep bandwidth, T_c is the chirp duration, and f_c is the frequency of carrier wave. Typical values are: $f_c = 60\,\mathrm{GHz}$, $B = 4\,\mathrm{GHz}$ and $T_c = 0.1\,\mathrm{ms}$. As a result, the transmitted radar chirp $s(t)$ is characterized by

$$s(t) = p(t)\mathrm{e}^{i2\pi\left(f_c t + (B/T_c)t^2/2\right)}.$$

$p(t)$ models the chirp envelope. Assuming a single target at distance d reflecting the transmitted chirp back, the time of flight τ is given by $\tau = 2d/c$ where c is the speed of the light. The reflection is mixed (i.e., multiplied) with the reference pulse $s(t)$ in the radar receiver, which can be modelized as

$$\begin{aligned} x(t) &\propto s^*(t-\tau) \times s(t) \\ &\propto p^*(t-\tau)p(t)\mathrm{e}^{i2\pi\left[f_c\tau - \frac{1}{2}\frac{B}{T_c}\tau^2 + \frac{B\tau}{T_c}t\right]} \\ &\approx \underbrace{p^*(t-\tau)p(t)}_{\approx |p(t)|^2}\mathrm{e}^{i2\pi f_c\tau} \cdot \mathrm{e}^{i2\pi\frac{B\tau}{T_c}t}, \end{aligned} \tag{1}$$

where $\frac{1}{2}\frac{B}{T_c}\tau^2$ is negligible and $p(t-\tau) \approx p(t)$ since $\tau \ll T_c$ (nanoseconds vs hundred microseconds) when the target distance d is smaller than $100\,\mathrm{m}$. Notice that the measured signal $x(t)$ is sinusoidal with a beat frequency given by $B\tau/T_c = 2Bd/(cT_c)$, thereby expressing the relation between frequency and range information.

In the radar community, it is common to ignore the envelope term $|p(t)|^2$ in (1) [3]. If we do so and assume that $x(t)$ contains one reflection at distance d, then capturing uniform point-wise measurements leads to a sequence of samples

$$x_n = A\mathrm{e}^{i\varphi}\mathrm{e}^{i\omega n}, n = 0, 1, \ldots, N-1. \tag{2}$$

$A \in \mathbb{R}_+$ is the magnitude. $\varphi \in]-\pi, \pi]$ is the phase, and

$$\omega = 4\pi\frac{Bd}{cT_c f_s}, \tag{3}$$

where f_s is the sampling frequency. This formulates the ranging problem (finding d) as a frequency estimation problem (finding ω), a classic problem in signal processing. In fact, assuming the frequency to have an imaginary part (i.e., $\Omega = \omega + i\alpha$ with $\alpha \in \mathbb{R}$) in this generic formulation, makes it possible to take into account the damping effect in $|p(t)|^2$ as we will see in this paper.

Estimate Small Displacements by Phase Difference: It is known that the signal phase φ in (2) is sensitive to the distance variations [4] and can be used to sense micro motions with μm-level accuracy [5]. When the displacement is within the range ambiguity (i.e. half of the carrier wavelength [4]), it can be estimated from the phase difference: let φ_1 and φ_2 be the phase before and after the displacement Δd, respectively. With some manipulations of (1), we have

$$\Delta d = \frac{c\Delta\tau}{2} = \frac{c(\varphi_2 - \varphi_1)}{4\pi f_c}. \tag{4}$$

Overcome the Barrier of Range Resolution: In the radar literature, the standard range resolution of an LFMCW radar is $c/(2B)$ which corresponds to the DFT resolution ($2\pi/\#\text{samples}$) in (2): for instance, the resolution of a typical consumer-level radar with bandwidth $B = 4\,\mathrm{GHz}$ is $3.75\,\mathrm{cm}$. Such a resolution doesn't always meet the accuracy requirement in real applications. For example, a centimeter-level ranging accuracy is desired for autonomous driving [6].

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To overcome this resolution barrier, it is necessary to super-resolve the frequency in the radar signal. Numerous model-based methods are available to this end (e.g. MUSIC [7], ESPRIT [8]). However, they are rarely applied in practice because of their lack of real-time properties [6] and their costly memory needs [9].

Super-resolving the frequency by interpolating on the DFT coefficients is generally much faster and has been explored extensively [10] [11] [12]. For example, Jacobsen et al [13] interpolated the frequency based on three DFT coefficients. Aboutanios et al [14] proposed an iterative estimator using two DFT coefficients. However, to the best of our knowledge, none of these estimators is exact: they are not able to compute the frequency even when given data is noiseless. Also, these estimators assume the frequency is real-valued, i.e., the signal has no damping effect. But many real-world signals (e.g. radar, NMR [15] or speech [16]) are damped.

This Paper: We propose a simple yet exact formula to estimate a single complex-valued frequency from just two DFT coefficients. In simulations we show that by iterating the formula for two times, the estimation uncertainty can achieve Cramér–Rao bound (CRB) [17], the optimal variance for unbiased estimates, for both damped and undamped sinusoid. We validate the formula on real radar measurements acquired in carefully designed experiments. We obtain promising accuracies for both the absolute distances estimated from the frequency, and for the small displacements estimated from the phase difference.

II. Formula for Estimating Single Frequency

We have shown in a recent paper that there exists a surprisingly exact formula that provides directly the nearly optimal relationship between the frequency of a single-frequency signal and two DFT coefficients [18]. In the sequel, we devise a new formula which, contrary to [18], accounts for a possible damping factor.

We define the Discrete Time Fourier Transform (DTFT) of an N-points discrete signal $\{x_0, x_1, \ldots, x_{N-1}\}$ by:

$$X(\omega) = \sum_{n=0}^{N-1} x_n \mathrm{e}^{-i\omega n}. \tag{5}$$

Theorem 1. *Assume that the samples x_n are of the form*

$$x_n = a_0 \mathrm{e}^{i\Omega_0 n}, n = 0, \ldots, N-1.$$

where $a_0 \in \mathbb{C}$ and $\Omega_0 \in \mathbb{C}$. Given two (real-valued) frequencies ω_1 and ω_2 such that $\omega_2 - \omega_1 = 2\pi/N$ and the DTFT X_1 and X_2 of x_n at ω_1 and ω_2, then the (complex-valued) frequency of the samples is given by

$$\Omega_0 = -i \ln\left(\mathrm{e}^{i\omega_1} \frac{X_1 - X_2}{X_1 - \mathrm{e}^{-i\frac{2\pi}{N}} X_2}\right). \tag{6}$$

Proof. Let $u_0 = \mathrm{e}^{i\Omega_0}$. Inserting $a_0 u_0^n$ in (5) and summing up the geometric sequence provides the analytic expression of the DTFT of the samples:

$$X(\omega) = a_0 \frac{u_0^N \mathrm{e}^{-iN\omega} - 1}{u_0 \mathrm{e}^{-i\omega} - 1}.$$

Given that $\omega_1 - \omega_2 = 2\pi/N$, we find that

$$X_1 = a_0 \frac{u_0^N \mathrm{e}^{-i\omega_1 N} - 1}{u_0 \mathrm{e}^{-i\omega_1} - 1},\ X_2 = a_0 \frac{u_0^N \mathrm{e}^{-i\omega_1 N} - 1}{u_0 \mathrm{e}^{-i(\omega_1 + 2\pi/N)} - 1}.$$

(same numerator) and so

$$\frac{X_1}{X_2} = \frac{u_0 \mathrm{e}^{-i\omega_1} \mathrm{e}^{-2i\pi/N} - 1}{u_0 \mathrm{e}^{-i\omega_1} - 1}.$$

Using basic algebra, we can finally extract

$$u_0 \mathrm{e}^{-i\omega_1} = \frac{X_1 - X_2}{X_1 - \mathrm{e}^{-2i\pi/N} X_2},$$

which is the same as (6). □

The L.H.S. of (6) is complex-valued (i.e.,$\Omega_0 = \omega_0 + i\alpha_0$) with its real part $\omega_0 \in]-\pi, \pi]$ being the frequency and its imaginary part $\alpha_0 \in \mathbb{R}$ being the damping factor. Of course, if we know that the singal has no damping effect, only the real-part of this formula needs to be considered. To use (6), we need a suitable choice of ω_1 and ω_2 to evaluate the DTFT. An effective way is to identify the peak ω_{DFT} of the DFT spectrum of the signal, and let $\omega_1 = \omega_{\text{DFT}} - \pi/N$ and $\omega_2 = \omega_{\text{DFT}} + \pi/N$.

Asymptotic Variance of the Formula Below we compute an asymptotic variance of (6) in a simplified scenario:

Proposition 1. *Consider a noise model*

$$y_n = a_0 \mathrm{e}^{i\Omega_0 n} + v_n, n = 0, \ldots, N-1$$

where the complex-valued samples v_n are additive white Gaussian noise with variance σ^2. Suppose

1) *the noise variance σ^2 is sufficiently small.*
2) *Ω_0 is real-valued, and*
3) *the formula (6) is applied with its input DTFT of y_n evaluated at $[\omega_1, \omega_2] = \Omega_0 + [-\pi/N, \pi/N]$.*

Denote the result of (6) as $\hat{\Omega}_0$ and the error of the real-valued frequency $\delta\omega_0 = \mathrm{Re}\{\hat{\Omega}_0\} - \Omega_0$. When N is sufficiently large,

$$\mathbb{E}[\delta\omega_0^2] = \underbrace{\frac{\pi^4}{96}}_{\approx 1.0147} \underbrace{\frac{12}{N^3 |a_0|^2/\sigma^2}}_{\text{CRB}}.$$

The proof is given in the appendix. The variance $\mathbb{E}[\delta\omega_0^2]$ is about 1% higher than the CRB. This shows (6) is nearly optimal even though only two DFT coefficients are used. The third condition of Proposition 1 can be attained by using the formula iteratively: let $\hat{\Omega}_0$ be the results of applying the formula once. Then we evaluate DTFT on $[\omega_1, \omega_2] = \mathrm{Re}\{\hat{\Omega}_0\} + [-\pi/N, \pi/N]$, respectively, and apply (6) again to obtain a refined estimation. As shown in Sec. III, one extra iteration is already nearly optimal.

III. Simulations

We validate (6) by running simulations to estimate single frequency from undamped sinusoid in Fig. 1 and damped sinusoid in Fig. 2. The root mean squared error (RMSE) of the estimated frequency ω_0 (i.e. the real part of Ω_0) is plotted together with CRB. With iterating twice, the formula of this paper overlaps CRB in both Fig. 1 and Fig. 2. This

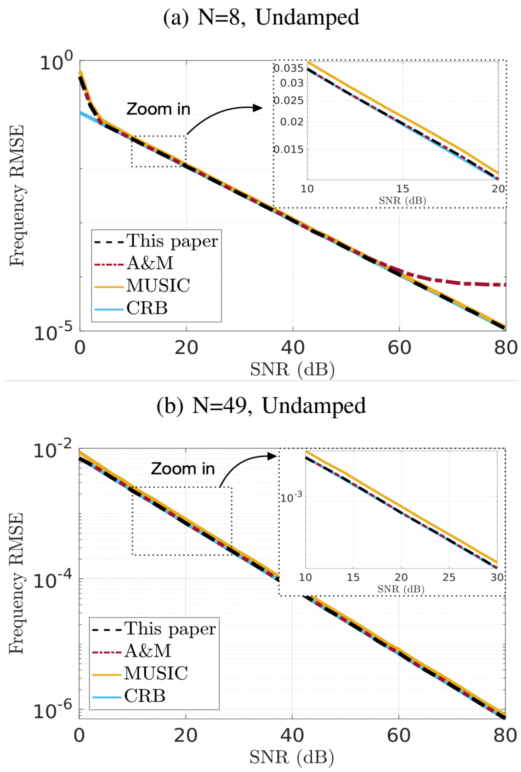


Fig. 1: Simulations for estimating single frequency from undamped sinusoid. 5000 noise realizations per SNR value. The formula of this paper and A&M formula are iterated twice.

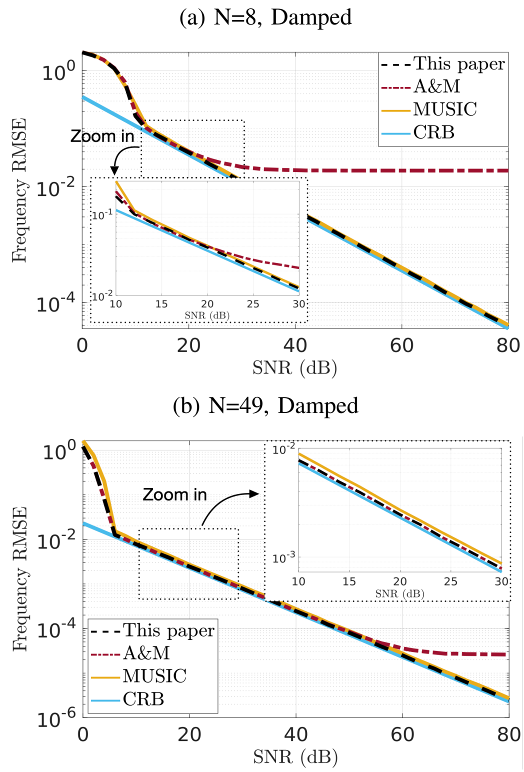


Fig. 2: Simulations for estimating single frequency from damped sinusoid. 5000 noise realizations per SNR value. The damping factor α is chosen such that the signal intensity decreases around 90% from start to end, i.e., $\mathrm{e}^{-\alpha N} \approx 0.1$.

indicates the formula (6) is nearly optimal no matter the signal is damped or undamped and no matter the number of samples N is small or large. A&M formula [14] with two iterations performs well for undamped signal with a large number of samples N (see Fig. 1b). But it degrades when N is small as shown in Fig. 1a, because A&M is not exact. A&M also diverges from CRB for damped signal as it assumes the signal has no damping effect. The subspace-based method MUSIC [7] is slightly above the CRB, but it is computationally expensive and is not suitable for real-time applications.

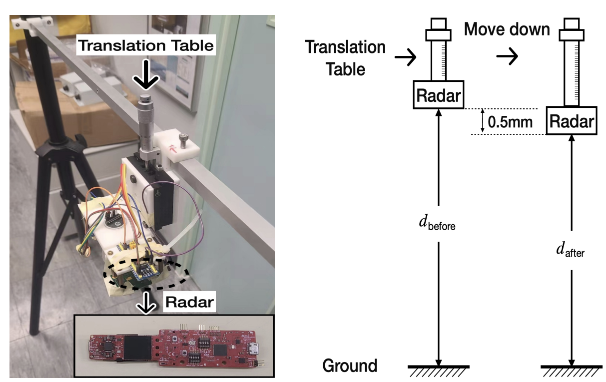
IV. Experiments

We validate our algorithm on real radar data. We use the low-cost mmWave sensor `IWR6843AOP` [19] from Texas Instrument. The radar data is processed using `Matlab 2021` on an i7-5930K processor with 64 GB memory.

A. Ranging Accuracy using Frequency

First, we evaluate the ranging accuracy of frequency by measuring the relative movement between the radar and the ground. As shown in Fig. 3, the radar is attached to a translation table, which makes it possible to move the radar vertically with high precision, typically by steps of $d_{\text{true}} = 0.5\,\text{mm}$. We collect measurements before and after the movement. Then we obtain the difference via $\hat{d} = d_{\text{after}} - d_{\text{before}}$ where d_{after} and d_{before} are computed from the estimation of frequency. We moved the radar 28 times and obtained 28 measurements of d_{true}. All radar data contains 512 samples. The nominal range resolution is $4.3\,\text{cm}$. The distance between the radar and the ground are around 0.9m.

We compare the RMSE of ranging results from different methods in Tab. I. Both the formula of this paper and the A&M formula are iterated two times. Our formula achieves an



(a) The radar is fixed at the bottom of the translation table, facing downwards.

(b) The 0.5mm movement is estimated by measuring the absolute distance d_{before} and d_{after}.

Fig. 3: Ranging single target from the frequency

TABLE I: Ranging single target from the frequency: RMSE of estimations and run time averaged over 28 measurements.

Methods	This paper	A&M	MUSIC
RMSE (mm)	0.250	0.250	0.335
Run time (ms)	0.2	0.2	280

average ranging error of $0.25\mathrm{mm}$ which is 172 times smaller than the nominal resolution. Meanwhile the computation only takes $0.2\mathrm{ms}$. This shows its great potential for real-time application. A&M method has a similar performance with our formula, probably because the radar signal has a large number of samples and its damping effect is not signifiant. The error of MUSIC is slightly worse. But MUSIC is computationally expensive: it takes $280\mathrm{ms}$ to estimate the frequency, which is more than 1000 times slower than using our formula.

B. Measuring Small Displacement using Phase Difference

Correct Phase Difference: Once the frequency estimation $\hat{\Omega}$ is available, the phase φ and the magnitude A are given by:

$$\hat{A}e^{i\hat{\varphi}} = \frac{1}{N}\sum_{n=0}^{N-1} x_n \mathrm{e}^{-i\hat{\Omega}n} = Ae^{i\varphi}\sum_{n=0}^{N-1}\mathrm{e}^{i(\omega-\hat{\Omega})n}/N, \quad (7)$$

where x_n denotes the samples of the radar signal in (2). Clearly, the frequency estimation error $\omega - \hat{\Omega}$ will impact the estimated phase. This impact can be corrected when we solve small displacements using the phase difference. For simplicity, assume $\hat{\Omega}$ is real-valued, i.e. $\hat{\Omega} = \hat{\omega} \in]-\pi, \pi]$. Then, from (7) we obtain:

$$\hat{A}e^{i\hat{\varphi}} = \frac{A}{N}\mathrm{e}^{i\left(\varphi+\frac{N-1}{2}(\omega-\hat{\omega})\right)}\frac{\sin\left(\frac{N(\omega-\hat{\omega})}{2}\right)}{\sin\left(\frac{\omega-\hat{\omega}}{2}\right)} \Rightarrow \hat{\varphi} = \varphi + \frac{N-1}{2}(\omega-\hat{\omega}).$$

Let $\{\varphi_1, \hat{\varphi}_1\}$ ($\{\omega_1, \hat{\omega}_1\}$) be the true and the estimated phase (frequency) before the small displacement Δd. And $\{\varphi_2, \hat{\varphi}_2\}$ ($\{\omega_2, \hat{\omega}_2\}$) be their counterparts after the displacement. Then the estimated phase difference is:

$$\hat{\varphi}_2 - \hat{\varphi}_1 = \varphi_2 - \varphi_1 + \frac{N-1}{2}(\omega_2 - \omega_1) + \frac{N-1}{2}(\hat{\omega}_1 - \hat{\omega}_2). \quad (8)$$

Both $\varphi_2 - \varphi_1$ and $\omega_2 - \omega_1$ are proportional to the displacement Δd. By inserting (3) and (4) into (8), we can compute Δd as:

$$\Delta d = \frac{c\left[\hat{\varphi}_2 - \hat{\varphi}_1 + \frac{N-1}{2}(\hat{\omega}_2 - \hat{\omega}_1)\right]}{2\pi\left[2f_c + \frac{(N-1)B}{f_s T_c}\right]}. \quad (9)$$

Experiment: Sensing micro motion using the phase can achieve $\mu\mathrm{m}$-level accuracy [4]. Hence, to evaluate the accuracy using the phase difference, we must devise an experiment which can provide a ground truth step size with $\mu\mathrm{m}$-level precision. Using the translation table in Sec. IV-A is not expected to have such an error-level because its minimum moving division is $10\mu\mathrm{m}$.

To provide a highly precise step size, we devise the following simple scheme:

TABLE II: Results of measuring small displacement using the phase difference. The units for all values are $\mu\mathrm{m}$. The RMSE of the estimated step size (from its nominal value) are averaged over 1K measurements per session.

Session	1	2	3	4	5	6	7
Step size (nominal)	9.5	10.3	11.3	9.0	11.7	13.0	11.5
RMSE of this paper	3.19	3.22	3.62	3.78	4.00	4.52	4.69

- The target (a small copper plate) is moved slowly towards the radar with a uniform speed v. Typically $v \approx 1\mathrm{cm/s}$.
- The radar measures the moving target 1K times uninterruptedly with an interval $T_m = 1\mathrm{ms}$. Hence, the whole measurement takes $1\mathrm{s}$.

The key of this experiment is to maintain a uniform speed when the target is moving. This is easy because the moving is slow ($\sim 1\mathrm{cm/s}$) and the moving duration is short. In particular, the target is moved by hand smoothly on a table for $3\mathrm{s}$. We record the moving length and compute the speed. The radar will measure the moving target during this 3s. The nominal step size is given by $vT_m \approx 10\mu\mathrm{m}$. Even when the true speed deviates from its estimation by 50% (which is too pessimistic), the error on the step size is $5\mu\mathrm{m}$, which is still not big.

In each session, the nominal step size (given by vT_m) is measured 1000 times. We measure seven sessions with different nominal step sizes and estimate it using (9). The ranging resolution is increased to $7.8\mathrm{cm}$ because we have to compromise the resolution for a shorter measurement interval T_m, which is $1\mathrm{ms}$ in our case. The distance between the radar and the target is around $0.8\mathrm{m}$.

Results: As shown in Tab. II, our formula obtains RMSE ranges from 3.19 to $4.69\mu\mathrm{m}$, which is more than 15 000 times smaller than the ranging resolution. This shows the great potential to apply our formula in precise industrial application such as vibration monitoring and machine calibration [20]. Other methods (A&M and MUSIC) achieves similar RMSE with a difference no more than tens of nanometers.

V. Conclusion

In this paper, we proposed a formula to estimate a single frequency using two DFT coefficients which is able to handle the damping effect. As demonstarted in simulations, the formula is nearly optimal in estimating frequency from both damped and undamped sinusoid. The formula also performs well on real radar data. For absolute distances computed from the frequency, the formula achieves an accuracy that is two orders of magnitude higher than the nominal resolution ($0.25\mathrm{mm}$ vs 4.3cm), and is significantly faster than the subspace-based method MUSIC. For small displacements computed from the phase differences, the obtained RMSE are more than four orders of magnitude smaller than the resolution.

Appendix: Proof of Proposition 1

Proof. The noisy DTFT at ω_1 is given by

$$\sum_{n=0}^{N-1} y_n \mathrm{e}^{-i\omega_1 n} = X_1 + \sum_{n=0}^{N-1} v_n \mathrm{e}^{-i\omega_1 n} = X_1 + \varepsilon_1,$$

where X_1 is the noiseless DTFT at ω_1, and ε_1 is the (complex-valued) noise with $\mathbb{E}[|\mathrm{Re}\{\varepsilon_1\}|^2] = \mathbb{E}[|\mathrm{Im}\{\varepsilon_1\}|^2] = N\sigma^2$. Let X_2 be the noiseless DTFT at ω_2 and ε_2 be the associated noise, we have

$$\begin{aligned}
i\cdot(\hat{\Omega}_0 - \Omega_0) =& \ln \frac{X_2+\varepsilon_2-X_1-\varepsilon_1}{(X_2+\varepsilon_2)\mathrm{e}^{-i\frac{2\pi}{N}} - X_1 - \varepsilon_1} - \ln \frac{X_2 - X_1}{X_2\mathrm{e}^{-i\frac{2\pi}{N}} - X_1} \\
=& \ln \frac{(X_2 - X_1 + (\varepsilon_2 - \varepsilon_1))\left(X_2\mathrm{e}^{-i\frac{2\pi}{N}} - X_1\right)}{\left(X_2\mathrm{e}^{-i\frac{2\pi}{N}} - X_1 + \varepsilon_2\mathrm{e}^{-i\frac{2\pi}{N}} - \varepsilon_1\right)(X_2 - X_1)} \\
=& \ln\left[\frac{X_2 - X_1 + (\varepsilon_2-\varepsilon_1)}{X_2 - X_1}\right] - \\
& \ln\left[\frac{X_2\mathrm{e}^{-i\frac{2\pi}{N}} - X_1 + \varepsilon_2\mathrm{e}^{-i\frac{2\pi}{N}} - \varepsilon_1}{X_2\mathrm{e}^{-i\frac{2\pi}{N}} - X_1}\right] \\
=& \ln\left[1 + \frac{\varepsilon_2 - \varepsilon_1}{X_2 - X_1}\right] - \ln\left[1 + \frac{\varepsilon_2\mathrm{e}^{-i\frac{2\pi}{N}} - \varepsilon_1}{X_2\mathrm{e}^{-i\frac{2\pi}{N}} - X_1}\right] \\
\approx& \frac{\varepsilon_2 - \varepsilon_1}{X_2 - X_1} - \frac{\varepsilon_2\mathrm{e}^{-i\frac{2\pi}{N}} - \varepsilon_1}{X_2\mathrm{e}^{-i\frac{2\pi}{N}} - X_1} \\
=& \underbrace{\frac{X_2\left(1 - \mathrm{e}^{-i\frac{2\pi}{N}}\right)}{(X_2 - X_1)\left(X_2\mathrm{e}^{-i\frac{2\pi}{N}} - X_1\right)}}_{f_1}\varepsilon_1 + \\
& \underbrace{\frac{X_1\left(\mathrm{e}^{-i\frac{2\pi}{N}} - 1\right)}{(X_2 - X_1)\left(X_2\mathrm{e}^{-i\frac{2\pi}{N}} - X_1\right)}}_{f_2}\varepsilon_2.
\end{aligned}$$

The error of the real-valued frequency $\delta\omega_0$ is given by

$$\begin{aligned}
\delta\omega_0 =& \frac{1}{2}\left[(\hat{\Omega}_0 - \Omega_0) + \overline{(\hat{\Omega}_0 - \Omega_0)}\right] \\
=& \mathrm{Re}\{f_1\}\mathrm{Im}\{\varepsilon_1\} + \mathrm{Im}\{f_1\}\mathrm{Re}\{\varepsilon_1\} + \\
& \mathrm{Re}\{f_2\}\mathrm{Im}\{\varepsilon_2\} + \mathrm{Im}\{f_2\}\mathrm{Re}\{\varepsilon_2\}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{E}\left[|\delta\omega_0|^2\right] =& \left(\mathrm{Re}\{f_1\}^2 + \mathrm{Im}\{f_1\}^2\right)N\sigma^2 + \\
& \left(\mathrm{Re}\{f_2\}^2 + \mathrm{Re}\{f_2\}^2\right)N\sigma^2 \\
=& \left(|f_1|^2 + |f_2|^2\right)N\sigma^2.
\end{aligned}$$

When N is sufficiently large,

$$\begin{aligned}
\lim_{N\to+\infty}|f_1| =& \lim_{N\to+\infty} \frac{|X_2|\cdot\left|1 - \mathrm{e}^{-i\frac{2\pi}{N}}\right|}{|X_1 - X_2|\cdot\left|X_2\mathrm{e}^{-i\frac{2\pi}{N}} - X_1\right|} \\
=& \frac{1}{|a_0|}\frac{\frac{2N}{\pi}\cdot\frac{2\pi}{N}}{\frac{4N}{\pi}\cdot\frac{4N}{\pi}} = \frac{1}{|a_0|}\frac{\pi^2}{4N^2}.
\end{aligned}$$

Note $|f_1| = |f_2|$ since $[\omega_1, \omega_2] = \Omega_0 + [-\pi/N, \pi/N]$. Hence,

$$\mathbb{E}\left[|\delta\omega_0|^2\right] = 2\frac{1}{|a_0|^2}\frac{\pi^4}{16N^4}N\sigma^2 = \frac{\pi^4}{96}\frac{12}{N^3|a_0|^2/\sigma^2}.$$

□

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