Fast and Robust High Resolution Frequency Estimation of Damped Signals

Qi Dai
Dept. of Electronic Engineering
The Chinese University of Hong Kong
Hong Kong, P.R.C.
daiq@link.cuhk.edu.hk

Ruiming Guo
Dept. of Electrical and Electronic Engineering
Imperial College London
London, United Kindom
ruiming.guo@imperial.ac.uk

Thierry Blu
Dept. of Electrical Engineering
National Taiwan University
Taipei, R.O.C.
thierry.blu@m4x.org

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***Abstract*—Estimating frequencies of damped sinusoids is the underlying problem of many practical applications. Existing algorithms are often inaccurate for estimating frequencies from damped sinusoids since many of them are designed for undamped signals. In this paper we propose an algorithm to estimate frequencies of a sum of damped sinusoids. This algorithm combines the Gauss-Newton method with an exact formula for single-frequency estimation. As validated by extensive simulations, the proposed algorithm provides nearly optimal estimations of frequencies of damped sinusoids, and is much more efficient than others when the signal has a large number of samples.**

***Index Terms*—Frequency estimation with high resolution, damped sinusoids, super-resolution.**

I. Introduction

Estimating the frequencies of a mixture of sinusoids, a classic topic in signal processing, is the underlying problem of many real-world applications. Typical applications include radar [1], communications [2] and NMR [3]. In this paper we address the estimation of frequencies from an N-point discrete signal sampled uniformly from a sum of sinusoids:

$$x_n = \sum\nolimits_{k=1}^{K} a_k \mathrm{e}^{i\Omega_k n}, n = 0, \ldots, N-1. \tag{1}$$

In this paper both the amplitudes a_k and the frequencies Ω_k are complex-valued. The imaginary part of Ω_k is known as the damping factor.

Existing Methods Many algorithms have been proposed to estimate frequency with high resolution. For example, in algorithms such as MUSIC [4], ESPRIT [5] and Matrix Pencil [6], a Toeplitz matrix is constructed by moving a window over the given N-point discrete signal, and the frequencies are estimated by exploring the structure of the Toeplitz matrix. More recently, algorithms based on compressive sensing (CS) [7] have been proposed for estimating frequencies. Typical algorithms include the on-grid L2-CS [8], the off-the-grid atomic norm minimization [9], and the Bayesian Superfast LSE method [10], etc. Another approach is to apply the iterative orthogonal matching pursuit (OMP) scheme to find the frequencies that best explain the given data in the sense of the maximum likelihood. The OMP-based algorithms differ in their way to estimate individual frequency. For example, in RELAX [11] the frequency is found by zero-padding; in Newtonized Orthogonal Matching Pursuit (NOMP) [12] the Newton method is used to minimize the residue energy and compute the frequency.

However, existing algorithms often have a poor performance when they are used in real applications, because real-world signals are often highly noisy and may deviate from the assumption of these algorithm. For example, many algorithms assume the frequencies Ω_k have a zero damping factor, i.e., $\mathrm{Im}\{\Omega_k\} = 0$, whereas many signals in practice (e.g., NMR [3], underwater acoustics [13] and speech [14]) are sufficiently damped for this hypothesis to be inaccurate. As shown later in our simulations, neglecting the presence of damping factors leads to a significantly degraded performance on estimated frequencies.

Besides discarding damping factors, most existing algorithms provide inaccurate estimations when frequencies are closely-spaced, i.e., when the separation between frequencies $|\mathrm{Re}\{\Omega_k\} - \mathrm{Re}\{\Omega_l\}|$ $(k \neq l)$ is small. Also, they are not efficient when the signal has a large number of samples.

Contributions In a previous paper [1] we gave an exact single frequency formula (3) which provides a nearly optimal estimation in the presence of noise. In this paper we propose an iterative algorithm based on that formula to estimate multiple frequencies with high resolution. The contributions of this paper are summarized below:

- We combine the formula (3) with the Gauss-Newton (GN) method to estimate multiple frequencies from a mixture of damped sinusoids.
- We propose an efficient estimator for multiple (complex-valued) frequencies. The proposed algorithm with this estimator is numerically stable when frequencies are closely spaced.
- We validate the accuracy and the efficiency of the proposed algorithm on extensive simulations. The proposed algorithm is shown to be faster and much more accurate than existing algorithms, especially for estimating closely-spaced frequencies.

This paper is organized as follows. In Sec. II we briefly review the formula (3), and then elaborate the proposed algorithms for estimating multiple frequencies. In Sec. III, we validate the proposed algorithm on extensive simulations. We conclude this paper in Sec. IV.

Gauss Newton

Project y_n on span $\left\{e^{i\hat{\Omega}_1 n}, \ldots, e^{i\hat{\Omega}_K n}, ne^{i\hat{\Omega}_1 n}, \ldots, ne^{i\hat{\Omega}_K n}\right\}$

$$\Updownarrow$$

$$y_n \simeq \sum_{k=1}^{K} (\hat{a}_k + n\hat{b}_k) e^{i\hat{\Omega}_k n}$$

$$\downarrow$$

Update $\hat{\Omega}_k$: applying the formula (3) on $(\hat{a}_k + n\hat{b}_k) e^{i\hat{\Omega}_k n}$

Fig. 1: Combining the Gauss-Newton method with (3).

II. Method

An Exact Formula for Single Frequency Estimation The Discrete-time Fourier Transform (DTFT) of an N-points discrete signal $\{x_0, x_1, \ldots, x_{N-1}\}$ is defined by:

$$X(\omega) = \sum\nolimits_{n=0}^{N-1} x_n e^{-i\omega n}. \tag{2}$$

Theorem 1. *Assume that the samples x_n are of the form*

$$x_n = a e^{i\Omega n}, n = 0, \ldots, N-1.$$

where $a \in \mathbb{C}$ and $\Omega \in \mathbb{C}$. Given two (real-valued) frequencies ω_1 and ω_2 such that $\omega_2 - \omega_1 = 2\pi/N$ and the DTFT X_1 and X_2 of x_n at ω_1 and ω_2, then the (complex-valued) frequency of the samples is given by

$$\Omega = -i \ln\left(e^{i\omega_1} \frac{X_1 - X_2}{X_1 - e^{-i\frac{2\pi}{N}} X_2} \right). \tag{3}$$

The proof is given in Appendix A. An effective choice of $[\omega_1, \omega_2]$ is to identify the peak ω_{DFT} of the DFT spectrum of the signal, and let $[\omega_1, \omega_2] = \omega_{\text{DFT}} \mp \pi/N$. In [1] we have shown, both theoretically and empirically, that (3) can be nearly as accurate as the Cramér-Rao bound [15].

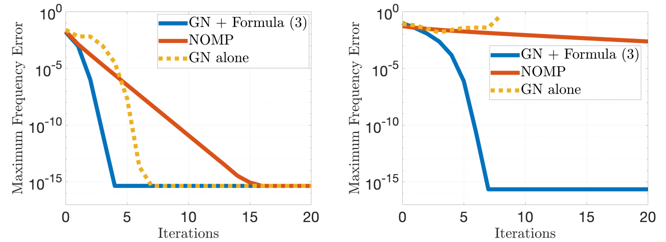
Estimating Multiple Frequencies Now we consider the estimation of K frequencies $\{\Omega_k\}_{k=1}^K$ from a sum of complex exponentials with additive noise v_n:

$$y_n = \sum\nolimits_{k=1}^{K} a_k e^{i\Omega_k n} + v_n, n = 0, \ldots, N-1. \tag{4}$$

The key of our algorithm is shown in Fig. 1. Assuming the availability of a previous estimation of the frequencies $\{\hat{\Omega}_k\}_{k=1}^K$: First, the coefficients $\{\hat{a}_k, \hat{b}_k\}_{k=1}^K$ on the basis $\{e^{i\hat{\Omega}_k n}, ne^{i\hat{\Omega}_k n}\}_{k=1}^K$ are calculated by least-squares fitting of the samples y_n. The rationale for using these two types of functions is the first-order Taylor approximation

$$e^{i(\hat{\Omega}_k + \delta\Omega_k)n} \approx e^{i\hat{\Omega}_k n} + i\delta\Omega_k \cdot ne^{i\hat{\Omega}_k n} \tag{5}$$

when $\delta\Omega_k$ is small—Gauss-Newton (GN) method. Second, for every frequency Ω_k, the DTFT of $(\hat{a}_k + n\hat{b}_k)e^{i\hat{\Omega}_k n}$ is evaluated at $\text{Re}\{\hat{\Omega}_k\} \pm \pi/N$, and the formula (3) is applied to update $\hat{\Omega}_k$. This process is repeated until convergence is reached. Note that, in contrast with the standard GN method, we do not update Ω_k directly using $\delta\Omega_k = -i\hat{b}_k/\hat{a}_k$ as suggested by (5), but indirectly using the N samples of $(\hat{a}_k + n\hat{b}_k)e^{i\hat{\Omega}_k n}$: this is



(a) $|\Omega_2 - \Omega_1| = 2 \times$ DFT bin (b) $|\Omega_2 - \Omega_1| = 0.5 \times$ DFT bin

Fig. 2: Plots of the estimation error of two frequencies Ω_1, Ω_2 in function of the number of iterations (no damping factor, no noise, 49 samples), demonstrating the effectiveness of our non-standard GN algorithm, in comparison to the standard GN version, and to NOMP, another iterative algorithm. Note: DFT bin $= 2\pi/N$ with $N = 49$.

significantly more robust as seen in Fig. 2b where the standard GN method fails to estimate two closely-spaced frequencies.

In conventional matching-pursuit based algorithms (e.g., NOMP), y_n is projected on the basis $\{e^{i\hat{\Omega}_k n}\}_{k=1}^K$. Our algorithm further employs the basis $\{ne^{i\hat{\Omega}_k n}\}_{k=1}^K$ which has the effect of accelerating convergence considerably as shown in Fig. 2. In contrast, the error of NOMP decreases slowly with the number of iterations, especially when $\{\Omega_1, \Omega_2\}$ are separated by less than a DFT bin (i.e., $2\pi/N$).

Initialization Our algorithm requires a set of initialized frequencies $\{\hat{\Omega}_k\}_{k=1}^K$ that are calculated successively as shown in Fig. 3. Again, in our initialization the GN method is combined with the formula (3).

Stopping Criterion The iteration of the GN-Formula method in Fig. 1 terminates when the Mean Squared Error (MSE) is sufficiently closed to MSE_1, where MSE is the error between y_n and its projection on the basis $\{e^{i\hat{\Omega}_k n}\}_{k=1}^K$; and MSE_1 is the error between y_n and its projection on $\{e^{i\hat{\Omega}_k n}, ne^{i\hat{\Omega}_k n}\}_{k=1}^K$. We know that when these two errors are equal, the MSE has reached a (local) extremum. In practice, we stop iterating when

$$\text{MSE}/\text{MSE}_1 \leq 1.001.$$

In most simulations, this condition is reached within 6 iterations. This indicates the fast convergence of our algorithm.

$\delta y_n = y_n$ $\rightarrow$

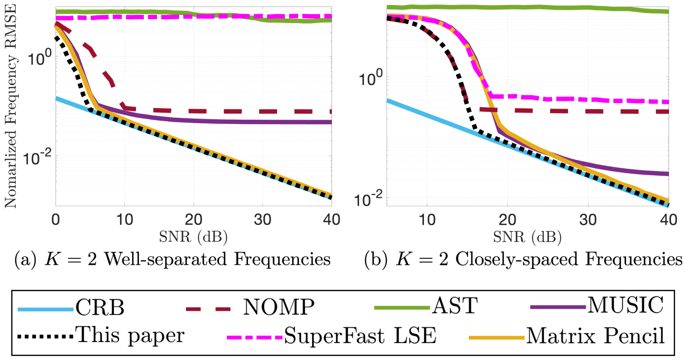
- Find the peak of $\text{DFT}(\delta y_n) \to \omega_{\text{DFT}}$;
- $[\omega_1, \omega_2] = \omega_{\text{DFT}} \pm \pi/N$;
- Evaluate DTFT & apply (3) $\to \hat{\Omega}_k$.

$\downarrow$

- Project y_n on span $\left\{e^{i\hat{\Omega}_1 n}, \ldots, e^{i\hat{\Omega}_k n}, ne^{i\hat{\Omega}_1 n}, \ldots, ne^{i\hat{\Omega}_k n}\right\}$;
- Remove this projection from $y_n \to \delta y_n$.

Repeat for all K frequencies

Fig. 3: Initialization of our algorithm.



(a) $K = 2$ Well-separated Frequencies (b) $K = 2$ Closely-spaced Frequencies

(c) Legends in (a) and (b)

Fig. 4: Plot of the frequency RMSE (see text for mathematical definition) in function of the noise level (2 complex-valued frequencies, 49 samples, additive white Gaussian noise): (a) frequency separation is 2 DFT bins; (b) frequency separation is half a DFT bin. Each value has been averaged over 5000 noise realizations. The damping factor is chosen so that the signal intensity decreases by about 50% from start to end.

4-DTFT Method When frequencies are closely-spaced, the algorithm described sometimes (albeit, infrequently) undergoes numerical instability, caused by the merging of these two frequencies. This issue can easily be detected due to a rise of the MSE. The MSE rises because the least-squares fitting of y_n becomes ill-conditioned when frequencies merge together, i.e., some basis in $\{\mathrm{e}^{i\hat{\Omega}_k n}\}_{k=1}^{K}$ are very similar.

To address it, we have devised a 4-DTFT method where two closely-spaced frequencies are estimated in by using four DTFT coefficients. The 4-DTFT method is a natural extension of the single-frequency formula (3) where two DTFT are used. The details of 4-DTFT method are given in Appendix B. We apply the 4-DTFT method only when the MSE increases. We check the MSE after each iteration of the whole GN-formula procedure (i.e., updating all K frequencies).

III. Simulations

We compare the accuracy and the efficiency of our algorithm with the state-of-the-art frequency estimation algorithms including MUSIC [4], Matrix Pencil [6], Atomic-norm Soft Thresholding (AST) [9], SuperFast LSE [10] and NOMP [12]. Of these methods, only Matrix Pencil does allow complex-valued frequencies. All other methods assume that the signals are un-damped. The implementation of these algorithms are either from the Matlab official, or provided by their respective authors. We omit the result of ESPRIT [5] since it has almost the same performance as Matrix Pencil.

Estimation Accuracy In Fig. 4 we compare the normalized RMSE of $K = 2$ frequencies estimations with the associated Cramér–Rao lower bound (CRB) [15], the optimal performance for an unbiased frequency estimator. The normalized RMSE is computed as:

$$\text{Normalized RMSE} = \frac{\sqrt{\mathbb{E}\left[\frac{1}{K}\sum_{k=1}^{K}\left(\mathrm{Re}\{\hat{\Omega}_k\} - \mathrm{Re}\{\Omega_k\}\right)^2\right]}}{2\pi/N}.$$

In this expression, the estimated frequencies $\hat{\Omega}_k$ are carefully paired with the true frequencies Ω_k. $\mathbb{E}[\cdot]$ denotes the expectation operator.

In Fig. 4(a) where the frequencies are well-separated, only our method is as accurate as the CRB. The RMSE of Matrix Pencil is slightly larger. All other methods are significantly less accurate. In Fig. 4b the frequencies are separated by only half of the DFT bin. The performance of Matrix Pencil degraded as its RMSE is always above the CRB. While the RMSE of our method is still nearly optimal when SNR is 17dB or above. Other methods are not accurate.

In Fig. 5 we plot the histogram of the estimation errors normalized by their respective CRB. The errors are collected from 10000 random tests where a noisy N-point signal is randomly generated and its frequencies are estimated. The setting of each test is given below:

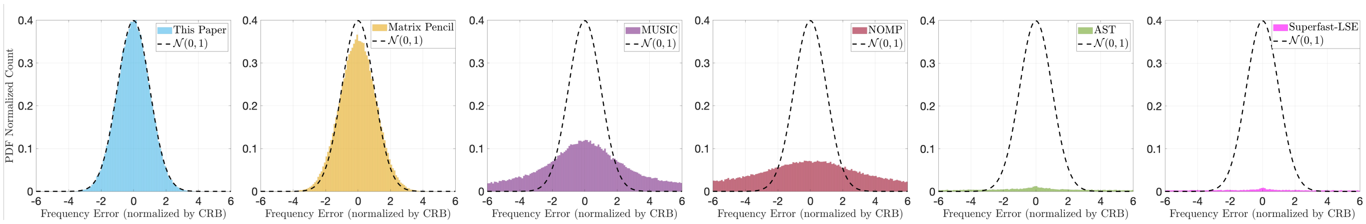
- N is randomly (uniformly) chosen between 49 and 127.
- Number of frequencies: $K = \lfloor N/5 \rfloor$;
- The SNR is chosen uniformly in the interval $[10, 40]$dB.
- $\{a_k, \Omega_k\}_{k=1}^{K}$ are randomly generated with the intensity of each sinusoid decreases between 50% to 90% from start to end, i.e., $0.1 \le \mathrm{e}^{-\mathrm{Im}\{\Omega_k\}N} \le 0.5$.
- The minimum separation between $\mathrm{Re}\{\Omega_k\}$ is $2 \times 2\pi/N$ in Fig. 5a and is $0.5 \sim 0.7 \times 2\pi/N$ in Fig. 5b.

The optimal histogram is expected to be that of the unit normal distribution $\mathcal{N}(0, 1)$ [16] (dashed line in Figs. 5). Only our method is able to produce a histogram close to the optimal distribution. Note, however, that our histogram plot in Fig. 5b is slightly below the $\mathcal{N}(0, 1)$-envelope. This is because of (very rare) initialization mistakes caused by missed frequencies, a problem that may arise when frequencies are too close and the signal is too noisy.

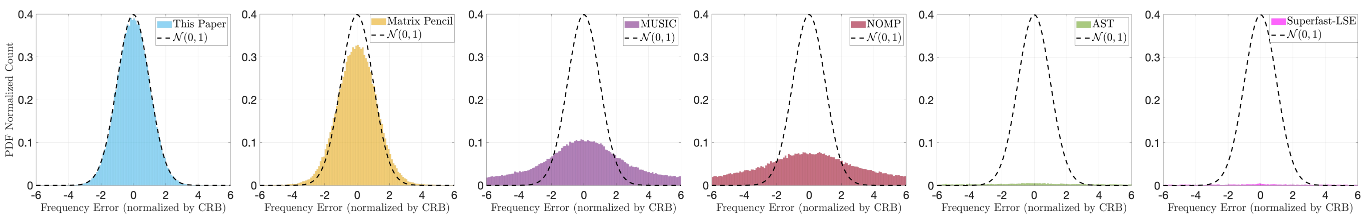
Run Time We also compare the run time of all the algorithms in Fig. 6. The simulations are done in Matlab 2024 on an i7-5930K processor with 64 GB memory. Overall, our method has the smallest run time when $N \geq 200$.

IV. Conclusion

In this paper, we propose an algorithm to estimate multiple frequencies from a sum of damped sinusoids. This algorithm combines an exact formula for single-frequency estimation with the Gauss-Newton method. We also propose a 4-DTFT method to address the numerical instability when frequencies are closely-spaced. The simulations show that our algorithm is able to accurately estimate multiple frequencies from a sum of damped sinusoids. And our method is comparatively even more efficient when the signal has a large number of samples.



(a) Well-separated frequencies. The minimum separation between frequencies $\mathrm{Re}\{\Omega_k\}$ in each test is $2\times$ DFT bin.



(b) Closely-spaced frequencies. The minimum separation between frequencies $\mathrm{Re}\{\Omega_k\}$ in each test is $0.5 \sim 0.7\times$ DFT bin.

Fig. 5: Histogram of frequency estimation errors from 10000 random tests (varying number of samples N, SNR, frequencies, amplitudes, noise, and varying number of frequencies: $K = \lfloor N/5 \rfloor$). The errors are normalized by their respective Cramér–Rao bound (CRB). When an algorithm is CRB-accurate, the statistics of its normalized errors are expected to be a unit normal distribution $\mathcal{N}(0,1)$.

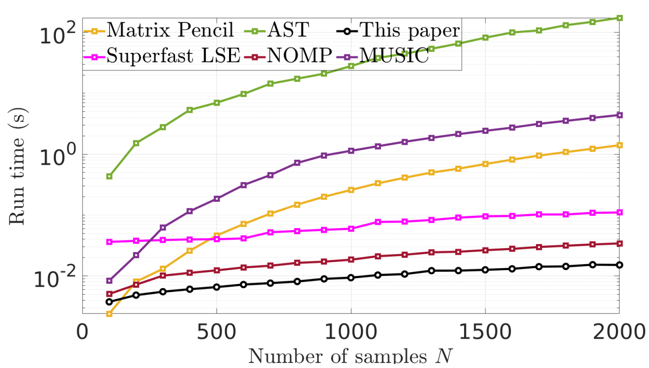


Fig. 6: Run time for solving $K = 16$ frequencies with the number of samples $N \in [100, 2000]$ with a step size of 100. The run time are averaged over 1000 random noisy realizations for each N. The SNR is fixed at 20dB.

Appendix A
Proof of Theorem 1

Proof. Given (2) and summing up the geometric sequence provides the analytic expression of the DTFT of the samples:

$$X(\omega) = a\frac{\mathrm{e}^{i\Omega N}\mathrm{e}^{-i\omega N} - 1}{\mathrm{e}^{i\Omega}\mathrm{e}^{-i\omega} - 1}.$$

Given that $\omega_1 - \omega_2 = 2\pi/N$, we find that

$$X_1 = a\frac{\mathrm{e}^{i\Omega N}\mathrm{e}^{-i\omega_1 N} - 1}{\mathrm{e}^{i\Omega}\mathrm{e}^{-i\omega_1} - 1},\ X_2 = a\frac{\mathrm{e}^{i\Omega N}\mathrm{e}^{-i\omega_1 N} - 1}{\mathrm{e}^{i\Omega}\mathrm{e}^{-i(\omega_1 + \frac{2\pi}{N})} - 1}.$$

(same numerator since $\omega_1 - \omega_2 = 2\pi/N$) and so

$$\frac{X_1}{X_2} = \frac{\mathrm{e}^{i\Omega}\mathrm{e}^{-i(\omega_1 + \frac{2\pi}{N})} - 1}{\mathrm{e}^{i\Omega}\mathrm{e}^{-i\omega_1} - 1}.$$

Using basic algebra, we can finally extract

$$\mathrm{e}^{i\Omega} = \mathrm{e}^{i\omega_1}\frac{X_1 - X_2}{X_1 - \mathrm{e}^{-i\frac{2\pi}{N}}X_2},$$

which is the same as (3). □

Appendix B
Details of the 4-DTFT Method

The 4-DTFT method exploits the fact that the DTFT of a sum of two sinusoids is a ratio between two polynomials. Consider the DTFT of $x_n = a_1\mathrm{e}^{i\Omega_1 n} + a_2\mathrm{e}^{i\Omega_2 n}$:

$$\begin{aligned} X\left(\omega + m\frac{2\pi}{N}\right) &= \frac{a_1(\mathrm{e}^{i(\Omega_1-\omega)N} - 1)}{\mathrm{e}^{i\Omega_1}\mathrm{e}^{-i(\omega+m\frac{2\pi}{N})} - 1} + \frac{a_2(\mathrm{e}^{i(\Omega_2-\omega)N} - 1)}{\mathrm{e}^{i\Omega_2}\mathrm{e}^{-i(\omega+m\frac{2\pi}{N})} - 1} \\ &= \frac{p_1 + p_2\mathrm{e}^{-i(\omega+m\frac{2\pi}{N})}}{1 + q_1\mathrm{e}^{-i(\omega+m\frac{2\pi}{N})} + q_2\mathrm{e}^{-i2(\omega+m\frac{2\pi}{N})}} \\ &= P\left(\mathrm{e}^{-i(\omega+m\frac{2\pi}{N})}\right) \Big/ Q\left(\mathrm{e}^{-i(\omega+m\frac{2\pi}{N})}\right). \end{aligned}$$

$P(z)$ and $Q(z)$ are a linear and a quadratic polynomial with coefficients $\{p_1, p_2\}$ and $\{1, q_1, q_2\}$, respectively. The roots of $Q(z)$ are given by $\{\mathrm{e}^{i\Omega_1}, \mathrm{e}^{i\Omega_2}\}$, i.e., the coefficients of $Q(z)$ form an annihilation filter from where the frequencies $\{\Omega_1, \Omega_2\}$ can be solved [17].

To solve $\{p_1, p_2, q_1, q_2\}$, a 4×4 linear system is constructed by evaluating 4 DTFT. We evaluate the DTFT at $\omega + [-3\pi/N, -\pi/N, \pi/N, 3\pi/N]$ with $\omega = (\mathrm{Re}\{\hat{\Omega}_1\} + \mathrm{Re}\{\hat{\Omega}_2\})/2$, and $\{\hat{\Omega}_1, \hat{\Omega}_2\}$ are some previous estimations of the frequencies. Then $\{\Omega_1, \Omega_2\}$ are solved by finding the roots of $Q(z) = 1 + q_1 z + q_2 z^2$.

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