An FOM for Gauging Numerical Synthesis Methods of Microwave Bandpass Filters

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***Abstract*—Various numerical methods for synthesizing microwave bandpass filters have been proposed in the past to explore new coupling topologies. However, the community lacks a figure of merit (FOM) for evaluating the superiority of the methods. To fill this gap and provide guidance for a better formulation, this letter will propose a general FOM for assessing the generality, completeness, and robustness of a numerical method when multiple solutions exist for a given coupling topology. The proposed FOM is based on the statistical distribution of the condition number of the Jacobian matrix. The performance of a numerical method is gauged by the convergence rate of the number of solutions. Different numerical solvers and ranges of initial values are also investigated to check the robustness of the FOM. Several well-received methods are studied, and their numerical performance is compared. In conclusion, the proposed FOM renders a general assessment for the formulations of numerical synthesis methods, which will lead to more powerful synthesis methods of microwave bandpass filters.**

***Index Terms*—Filter synthesis, numerical methods, performance evaluation.**

I. Introduction

THE synthesis of microwave bandpass filters has always been an important topic to accommodate various needs in practical filter designs. The increasing demands for more compact structures and simpler layouts suitable for high volume production spur the research on numerical synthesis approaches of coupled resonator bandpass filters, which may possess irregular coupling topologies for nearly arbitrary filtering characteristics. For such applications, the exhaustive synthesis becomes the sole option for advanced bandpass filters, whose topologies come with multiple solutions.

Classical filter synthesis methods depend on the known sequences of Given's orthogonal transformation from the coupling matrix in a well-known canonical form, such as the folded or arrow form [1], to that in the desired topology. The methods have played pivot role in synthesizing many practical bandpass filters with a handful of coupling topologies, such as cascaded-trisection (CT), cascaded-quadruplet (CQ), and extended-box topologies. However, the intuitively found transformation "recipes" for these coupling topologies lack generality for other practical coupling topologies whose recipes are not known, not mentioning that the multiple solutions always come with the irregular coupling topologies.

On the other hand, numerical synthesis method inevitably presents the future trend in synthesizing coupling matrices with irregular coupling topologies when the "recipes" are not available. Over the past few decades, various numerical methods based on different formulations that associate the coupling matrix $\mathbf{M}$ in a given coupling topology with the objective function have been proposed. Directly searching for $\mathbf{M}$ is one of the popular schemes, in which the objective functions are defined by the least squares error (LSE) of the magnitude of S_{11} and S_{12} at the reflection zeros (RZs) and transmission zeros (TZs) [2], [3]. Such a scheme constructs the mapping from the $\mathbf{M}$ domain to scattering parameters domain $\mathbf{S}$ through a highly nonlinear relation. In [4], the coefficients of polynomials (E, P, F) of the filtering function are utilized as objective function.

Another popularly used formulation is to numerically find the required orthogonal matrix $\mathbf{Q}$ that transforms the coupling matrix $\mathbf{M}$ in a canonical form to that in the desired topology. Although a semianalytical approach was proposed two decades ago based on Gröbner basis method [5], its complexity hinders applications to high-order filters. Nevertheless, its numerical version utilizes isospectral flow method has shown its potential for high-order applications [6], [7]. Recently, a numerical exhaustive synthesis framework has been put into use, whose formulation maps the desired $\mathbf{M}$ to the target coupling matrix $\mathbf{F}$ in the canonical folded form that can be obtained *uniquely* from the given filtering function [8]. The method can obtain all the real-valued solutions for any viable coupling topology.

Having had various numerical synthesis methods, a vital question arises: is there any figure of merit (FOM) that can describe the superiority of the formulation of a numerical synthesis method? This question is fundamental in twofold: 1) to reflect the legitimacy of the coupling topology for the desired $\mathbf{M}$; and 2) to indicate the numerical efficiency for searching all real-valued solutions. Here, the legitimacy means that the topology can provide finite number of solutions for the given filtering function, real- or complex-valued.

In this letter, the question is addressed by proposing a FOM for comparing numerical synthesis methods. The numerical process is assumed to be a solution solver rather than an optimizer as an optimal solution is sometimes subjective. An objective FOM should reflect two aspects of a synthesis approach that solely depends on the formulation: the legitimacy identification ability and numerical efficiency. The FOM proposed is defined by two statistical characteristics: 1) the mean value of the common logarithm of the condition number of pseudo Jacobian matrix $\mathbf{J}^{\mathrm{T}}\mathbf{J}$ and 2) the successful convergence rate versus the condition number. It will be shown that with the FOM, the completeness for all solutions, and the robustness and efficiency of a gradient-based solution solver can be clearly revealed. Here, $\mathbf{J}$ is the Jacobian matrix that associates the variable vector $\mathbf{x}$ of $\mathbf{M}$ to target vector $\mathbf{F}_0$, and the condition number refers to the ratio of the maximum and minimum eigen-values of a square matrix. Except the

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numerical approaches with a square matrix **J**, approaches with nonsquare **J** lose the ability of identifying the legitimacy. It will be demonstrated that having a low value of condition number and high successful convergence rate is particularly important for finding all solutions of **M**.

For better illustration, four different formulations are investigated, including: 1) mapping **M** to **F** in the folded form; 2) mapping **M** to the coefficients of $E(s)$, $P(s)$, and $F(s)$ of the filtering function; 3) mapping **M** to the RZs and TZs of the filtering function; and 4) mapping the matrix **Q** to null entries of **M**. Among them, only the formulations that come with a square Jacobian matrix can justify the legitimacy of the topology. It will be shown that the probability density function (pdf) of the condition numbers of pseudo-Jacobian matrix and the successful convergence rate for different formulations dominate the numerical performance significantly, whereas the numerical solvers and ranges of initial values affect the performance insignificantly. The following two conclusions can be drawn: 1) the ability to reveal legitimacy depends on the formulation and 2) the numerical performance, as described by the FOM, also heavily depends on the formulation, rather than the numerical solvers nor the initial values. It is the first time to have a general FOM for gauging a numerical synthesis method and providing guidance to find a better numerical synthesis approach.

II. Mathematical Background

According to the recent study on exhaustive synthesis [8] and inverse function theorem [9], the legitimacy of a coupling topology can be identified by the nonzero determinant of the Jacobian matrix **J** of the mapping from **M** to **F** is nonzero in a vicinity of a trial solution, where **M** and **F** contain N_M and N_F variables, respectively, and $N_M = N_F$. Therefore, to be able to identify legitimacy of a coupling topology, the Jacobian matrix **J** of the formulation must be a square matrix.

Considering a traditional Gauss–Newton solver and assuming the **J** of the formulation is not square, the searching direction and step length for updating variable vector **x** to minimize $\|\mathbf{F}(\mathbf{x}) - \mathbf{F}_0\|$ is found by

$$d\mathbf{x} = -\left(\mathbf{J}^{\mathrm{T}}\mathbf{J}\right)^{-1}\mathbf{J}^{T}\left(\mathbf{F}(\mathbf{x}) - \mathbf{F}_0\right) \tag{1}$$

where $(\mathbf{J}^{\mathrm{T}}\mathbf{J})^{-1}\mathbf{J}^{T}$ is the pseudo-inverse of the Jacobian matrix, $\mathbf{F}(\mathbf{x})$ is the vector mapping function, and $\mathbf{F}_0$ is the target function value. For other solution solvers, such as the Levenberg–Marquardt algorithm (LMA) [10], the modified updating step is given by

$$d\mathbf{x} = -\left(\mathbf{J}^{\mathrm{T}}\mathbf{J} + \lambda\mathbf{I}\right)^{-1}\mathbf{J}^{T}\left(\mathbf{F}(\mathbf{x}) - \mathbf{F}_0\right) \tag{2}$$

which combines the traditional Gauss–Newton and the steepest descent methods in the sense of gradient-based optimization. Naturally speaking, the Gauss–Newton direction is the dominant component in searching for a solution. The steepest decent portion is just for complementing the searching process when the Jacobian matrix is singular. Note that the $(\mathbf{J}^{\mathrm{T}}\mathbf{J})^{-1}$ can only be calculated when the $\mathbf{J}^{\mathrm{T}}\mathbf{J}$ is nonsingular. In other words, the singularity, or the condition number, of the matrix $\mathbf{J}^{\mathrm{T}}\mathbf{J}$ decides the performance of a searching process.

Considering the **M** for a given topology incorporating multiple solutions in a finite range, the statistical distribution of the condition numbers of matrix $\mathbf{J}^{\mathrm{T}}\mathbf{J}$ can reveal the nature of the solution process. For a chosen formulation, based on its objective function, the samples of the condition numbers of matrix $\mathbf{J}^{\mathrm{T}}\mathbf{J}$ are generated by uniformly sampling the initial values; then, the approximate pdf of the condition number is obtained. If the mean value of common logarithm of condition numbers is too large, over ten for instance, the solution search process will be easily trapped.

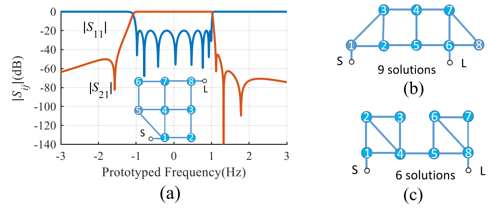

Fig. 1. (a) 8-4 filter response and grid topology; (b) 8-4 trapezoid topology; (c) 8-4 CQ topology.

Additionally, the successful convergence rate also needs to be investigated. The rate is measured by the ratio of the number of trials that successfully find solutions over the total number of trials.

III. Numerical Synthesis Formulations

This section reviews several known numerical synthesis formulations, including the mapping of **M** to **F**, mapping of **M** to the coefficients of $E(s)$, $P(s)$, and $F(s)$, mapping of **M** to the RZs and TZs of **S**, and mapping of **M** to orthogonal transformation matrix **Q**. The statistical distribution of the condition numbers for each of the formulations are investigated and compared together with the successful convergence rate to benchmark the numerical synthesis formulations. For comparison purposes, the 8-4 filter in the grid topology associated with the asymmetrical target filter response shown in Fig. 1(a) is chosen as a benchmark example, which results in total 18 real-valued solutions. The initial values are uniformly sampled in the range of $[-1, 1]$. The LMA is used with the same setting and convergence threshold of 10^{-10}. The trapezoid and CQ topologies shown in Fig. 1(b) and (c) are also testified for demonstrating the generality of the proposed FOM.

A. Mapping of M to F

As shown in [8], the coupling matrix **F** in the folded form has two attractive features: 1) it is unique for the given filtering function and 2) a coupling matrix in any topology can be transformed to **F** with a known sequence of orthogonal transformations. These two features ensure **F** to be an ideal target for mapping a coupling matrix **M** in a desired topology to a unique **F**, establishing a "*comparing apples to apples*" mapping relation. The mapping formulation can be implicitly described by

$$\begin{aligned}\mathbf{F}_0(1) &= f_1\left(\mathbf{M}(1), \mathbf{M}(2), \ldots, \mathbf{M}(N_M)\right)\\ \mathbf{F}_0(2) &= f_2\left(\mathbf{M}(1), \mathbf{M}(2), \ldots, \mathbf{M}(N_M)\right)\\ &\vdots\\ \mathbf{F}_0(N_F) &= f_{N_F}\left(\mathbf{M}(1), \mathbf{M}(2), \ldots, \mathbf{M}(N_M)\right)\end{aligned} \tag{3}$$

where $\mathbf{F}_0$ is the target folded coupling matrix directly obtained from a filtering function. The legitimacy of the given topology for **M** can be checked through the rules that $N_M = N_F$ and nonzero determinant of **J** in the solution domain. All the real-valued solutions can be numerically found through solving (3) by minimizing $\|\mathbf{F}_0 - f(\mathbf{M})\|$ below the convergence threshold.

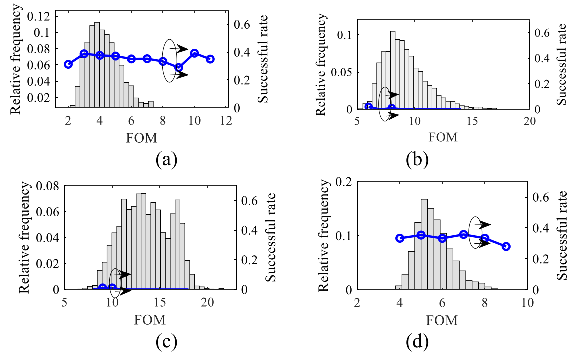

Fig. 2. Relative frequency histogram of FOM and successful convergence rate for mappings of (a) **M** to **F**; (b) **M** to $E(s)$, $P(s)$ and $F(s)$: (c) **M** to RZs and TZs of S-parameters; (d) **Q** to null entries of **M.**

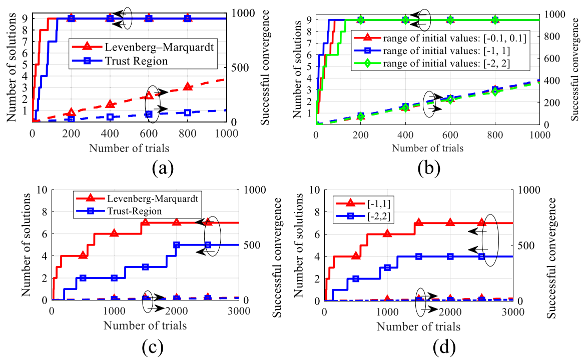

Fig. 3. (a) Convergence of **M** to **F** versus different numerical solvers. (b) Convergence of **M** to **F** versus ranges of initial values. (c) Convergence of **M** to (*E, P, F*) versus different numerical solvers. (d) Convergence of **M** to (*E, P, F*) versus ranges of initial values.

The relative frequency histogram of the common logarithm of the condition number of $\mathbf{J}^{\mathrm{T}}\mathbf{J}$ for 1000 random initial values, the approximate pdf, for synthesizing the filter is shown in Fig. 2(a), which is centered around four. Most of the common logarithm of condition numbers are below eight. The distribution for successful convergence rate with respect to different ranges of the condition number is also presented in Fig. 2(a), showing around 35% successful rate. Total 18 solutions can be found in less than 700 trials, taking less than 10 s on an ordinary PC.

B. Mapping of M to (E, P, F) or S

Assuming that the coefficients of polynomials (*E, P, F*) are known in the filter synthesis prior.

As discussed in [12], N_M equals to the number of variables in P(s) and F(s) so that legitimacy of the coupling topology can be justified in mapping **M** to the coefficients of the polynomials, the pdf of the condition number of which for 1000 random initial values, and the successful convergence rate are superimposed in Fig. 2(b), showing the mean value of the condition numbers is five orders of magnitude larger than that of mapping **M** to **F** and the successful convergence rate is less than 1%. The convergence process states that only 7 out of 18 distinct solutions are found after 3000 trials.

Similar to the formulation of mapping **M** to the coefficients, mapping **M** to the nulls of S-parameters at the RZs and TZs of the target filtering function also exhibits a poor pdf of the condition number and low successful convergence rate. As shown in Fig. 2(c), a cluster of common logarithm of condition numbers are below eight located above ten, which interrupts the low convergence rate of less than 1%. Only 3 out of 18 solutions are found by 3000 trials. Such a synthesis formulation with a poor FOM will face difficulties to find all the solutions.

TABLE I

FOM FOR 8-4 TOPOLOGIES WITH DIFFERENT FORMULATIONS

Formulations	Mean Log. of Cond. Number			Mean Successful Rate (%)			Legitimacy Identify
	Grid	CQ	Trape.	Grid	CQ	Trape.	
M to **F**	4.39	5.06	4.88	35	87	42	✓
M to (E, P, F)	9.19	9.75	9.12	0.5	0.8	0.3	✓
M to $\lvert S_{ij}\rvert$ at RZs and TZs	13.56	14.58	13.86	0.1	0.1	0.2	✗
Q to Null entries of **M**	5.62	5.04	5.66	34	96	38	✗

C. Mapping of Q–M

Numerically finding the orthogonal matrix **Q** that transforms the target coupling matrix $\mathbf{F}_0$ in a canonical form to **M** in the desired topology can be formulated as

$$\left\{\mathbf{Q}\mathbf{F}_0\mathbf{Q}^T\right\}_{i,j} = 0 \quad \forall\,(i,j) \in \{(i,j)\,,\ i \neq j : \mathbf{M}(i,j) = 0\} \qquad (4)$$

with $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$.

The tricky part of this formulation is how to locate the nonzero elements in the unknown transformation matrix **Q** intuitively subject to the coupling topology of **M**. Since the number of variables in **Q** is usually not equal to the number of equations in (4), the formulation is not guaranteed to form a square Jacobian matrix **J**; therefore, lacking the ability to identify legitimacy. For example, the **Q** for the 8-4 grid filter has 30 variables, not equal to the number of equations of 46 in (4). Being **J** not a square matrix, a given coupling topology is not guaranteed to have finite number of solutions, which ensure physical realizability. The pdf of the condition number and the successful convergence rate are presented in Fig. 2(d), which is comparable to that of mapping **M** to **F** as the formulation essentially is another *comparing apples to apples* case.

D. Dependence of Numerical Solvers and Initial Values

Numerical solvers and the ranges of initial values for a numerical synthesis method play a less critical role if the formulation is superior. Taking the 8-4 trapezoid filter [13] with the response shown in Fig. 1, which has total nine real solutions, as an example. Two numerical solvers, which are Levenberg–Marquardt and trust-region algorithms, and three different ranges of initial values are applied to the formulations of **M** to **F** and **M** to (*E, P, F*). As demonstrated in Fig. 3(a) and (b), the successful convergence rate of the **M** to **F** formulation is not sensitive to the numerical solvers and the range of initial values. However, when different solvers are used to the **M** to (*E, P, F*) formulation, as shown in Fig. 3(c), the convergence rate is sensitive to the solvers, and the successful rate is still inferior. The different ranges of initial values are also implemented for **M** to (*E, P, F*) formulation. Similar results can be observed in Fig. 3(d).

E. Comparison of FOM of Numerical Synthesis Methods

In Table I, the FOM for three different 8-4 coupling topologies with four different formulations of numerical synthesis methods are listed. Among all the three coupling topologies, only the mappings from $\mathbf{M}$ to $\mathbf{F}$, and $\mathbf{Q}$ to null entries of $\mathbf{M}$ accomplish exhaustive search for all solutions within 3000 trials. Other two formulations fail due to the poor successful convergence rate. It is clear that for a given coupling topology, a poor FOM leads to poor convergence and failure of exhaustive solution search. The conclusion can be drawn that the mean value of common logarithm of condition number of $\mathbf{J}^{\mathrm{T}}\mathbf{J}$ is a critical gauging parameter that determines the successful convergence rate of a numerical synthesis method of microwave bandpass filters. In other words, the formulation decisively determines the completeness for all solutions, the robustness and efficiency of a numerical synthesis approach, whereas the coupling topologies, numerical solvers, and the initial values are less critical. Among four popular formulations for synthesizing coupling matrix $\mathbf{M}$ for a given coupling topology and filtering function, the mapping of $\mathbf{M}$ to $\mathbf{F}$ is the only one that renders a complete set of solutions with a high numerical efficiency and the ability to identify legitimacy of the coupling topology.

IV. Conclusion

An objective FOM for gauging the performance of a numerical synthesis method is a piece of missing parts in the domain of microwave filter synthesis. This work proposes a FOM for evaluation of a numerical synthesis method in two aspects: 1) finding the statistic mean value of the common logarithm of the condition number of $\mathbf{J}^{\mathrm{T}}\mathbf{J}$, which is solely determined by the formulation of a method and 2) the successful convergence rate. The ability to identify legitimacy of the coupling topology is also considered as a critical feature. As demonstrated by intensive numerical case studies, the formulation of mapping $\mathbf{M}$ to $\mathbf{F}$ is superior to other existing formulations. The FOM can provide a guidance to find more efficient numerical synthesis methods for exhaustive synthesis of microwave coupled resonator bandpass filters.

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